An Overview of Simplicial and Singular Homology Groups

Ethan Roy

December 2020

1 Introduction

These notes provide an insight to the simplicial and singular homology groups of topological spaces as well as one of their applications in the Euler Formula. We start by defining the simplicial homology, a relatively easy way to find homology groups for topological spaces that can be triangulated. We then show how our definitions of the simplicial homology groups are used by looking at the homology groups of the Torus using our definied simplicial homology theory. The simplicial homology theory has its limits though, so we then analyze a slightly "weirder" way to calculate homology groups through singular homology theory. Singular homology theory lets us find homology groups for all topological spaces regardless if they can be triangulated or not. We finally end the paper with a look at the Euler-Poincaré formula and how its application of Homology groups is important for comparing geometric objects.

These notes are primarily based on the content of Richard J. Trudeau's *Introduction to Graph Theory* as well as Fred H. Croom's *Basic Concepts of Algebraic Topology.*

2 Pre-requisites

Previous knowledge of kernels, images, homeomorphisms, and isomorphisms is assumed for this report

This paper relies on a couple definitions from group theory. We first start by defining a group G as a set such that for any two elements $a, b \in G$ there exists an operation on a, b such that said operation produces another element, $p \in G$. Let us call this operation $\sim$. In order for G to be a group it must satisfy the following properties:

1. The operation must be associative. Meaning that for $a, b, c \in G$ we must have that $(a \sim b)\ c = a \sim (b \sim c)$

2. There must also exist an identity element. Meaning that for all $a \in G$ there must exist an $e \in G$ such that $e \sim a = a$ and $a \sim e = a$

3. Each element in G must contain an inverse. Meaning that for an identity element $e \in G$, and for some $a \in G$ there must also be a $a^{-1} \in G$ such that $a \sim a^{-1} = a^{-1} \sim a = e$

We then call a subset H of a group G a **subgroup** if it has the following properties:

1. Closure: If $a \in H$ and $b \in H$ then $ab \in H$

2. Identity: $1 \in H$

3. Inverse: If $a \in H$ then $a^{-1} \in H$

For any subgroup H for a group G we define a **left coset** as a subset of H such that:
$aH = \{ah | h \in H\}$

We say a a subgroup N for a group G is a **normal subgroup** if and only if $gng^{-1} \in N$ for all $g \in G$ and $n \in N$

Finally for a group G and a normal subgroup N of G we define a **quotient group**, G/N, as the set of cosets of N in G.

3 Simplicial Homology Groups

3.1 Definitions

We start by showing what a set of distinct points looks like in a higher dimension Euclidiean space. We say that a set of $k+1$ points, $\{a_0, ..., a_k\}$ is **geometrically independent** if there is no hyperplane of dimension $k-1$ that contains all of the points. Meaning for our set $\{a_0, ..., a_k\}$ no three points lie on a line, and no four of them are contained in a plane, and so on and so forth.

We move on by properly defining what our shapes are going to look like for our geometrically independent points in these higher dimensions. For a set of k geometrically independent points, $\{a_0, ..., a_k\}$ in $\mathbb{R}^n$ for $k \leq n$, we define the **k-dimensional geometric simplex**, or **k-simplex**, σ^k, as the set of all points $x \in \mathbb{R}^n$ such that there exists non-negative $\lambda_0, ..., \lambda_k \in \mathbb{R}$ where:

1. $x = \Sigma_{i=0}^{k} \lambda_i a_i$

2. $\Sigma_{i=0}^{k} \lambda_i = 1$

We say that the $\{a_0, ..., a_k\}$ are vertices of σ^k and σ^k is positively oriented, $+\sigma^k$, for the class of even permutations of the vertices and that σ^k is negatively oriented, $-\sigma^k$, for the class of odd permutations of the vertices.

An example of an oriented $k - simplex$ would be if we look at a $\sigma^1 = \{a_0, a_1\}$ simplex, an edge, then an example of the positive orientation would be $+\sigma^1 = \{a_0, a_1\}$ and an example of the negative orientation would be $-\sigma^1 = \{a_1, a_0\}$.

When looking at simplexes we also want to analyze how they interact, so we say two simplexes, σ^m and σ^n are properly joined if they do not intersect or if $\sigma^m \cap \sigma^n$ is a face for both simplexes.

For non-negative integers k, n such that $k \leq n$, we say that a simplex σ^k is a face for another simplex σ^n if for all $a_i \in \sigma^k$ we have $a_i \in \sigma^n$.

For a topological space X we look at the triangulation of said space in order to then define a **complex**, K, as a finite family of properly joined k-simplexes such that K contains the faces of each of the $k - simplexes$. The dimension of K is the largest positive integer k of all the k-simplexes in K. We call K an oriented complex if each of its k-simplexes is assigned an orientation.

Suppose for a positive number p we have simplexes σ^p and σ^{p+1}. Then for the pair (σ^p, σ^{p+1}) we say the **incidence number**, denoted $[\sigma^{p+1}, \sigma^p]$, as $[\sigma^{p+1}, \sigma^p] = 0$ if σ^p is not a face for σ^{p+1}. If σ^p is a face for σ^{p+1} then there exits a vertex, $v \in \sigma^{p+1}$ and $v \notin \sigma^p$, such that we can write
$+\sigma^{p+1} = \pm\{va_0, a_1, ..., a_p\}$. If $+\sigma^{p+1} = +\{va_0, a_1, ..., a_p\}$ then $[\sigma^{p+1}, \sigma^p] = 1$ and if $+\sigma^{p+1} = -\{va_0, a_1, ..., a_p\}$ then $[\sigma^{p+1}, \sigma^p] = -1$.

Now that we know more about complexes, simplexes, and orientation we can move onto to defining the components for a simplicial homology group. We start with a positive number p, we then define a $p - chain$ as a function c_p from a family of oriented $p - simplexes$ from a complex K such that for each p-simplex, σ^p, we have that $c(-\sigma^p) = -c(+\sigma^p)$. For example, a $0 - chain$ would be a function from the $0 - simplexes$ to the integers. We denote the $p - chain$ group, $C_p(K)$,from the family of $p - chains$ on a complex K.

We say an **elementary p-chain** is a $p - chain$, $c(\sigma^p)$ such that $c(\tau^p) = 0$ where τ^p represents every simplex different from σ^p. We denote an elementary p-chain by $g \cdot \sigma^p$ where $g = c_p(+\sigma^p)$. Also note that any $p - chain$, d_p, can be expressed as an arbitrary sum, $d_p = \Sigma g_i \cdot \sigma^p$, of elementary p-chains.

Suppose we have an elementary p-chain, $g \cdot \sigma^p$, for a positive integer p. Then we say the **boundary** of $g \cdot \sigma^p$, denoted as $\partial(g \cdot \sigma^p)$, as

$$\partial(g \cdot \sigma^p) = \Sigma[\sigma^p, \sigma_i^{p-1}]g \cdot \sigma_i^{p-1} \tag{1}$$

for a family of σ_i^{p-1} simplexs in a complex K.

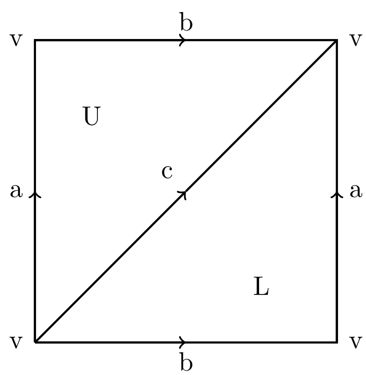
Suppose we have a complex, K, with some orientation imposed on it. Then we define the **p-cycle** as a *p-chain*, z_p such that $\partial(z_p) = 0$. We denote the **p-dimensional cycle group**, $Z_p(K)$, as a family of these p-cycles otherwise a homeomorphism of the kernel of the boundary mapping, $\partial : C_p(K) \to C_{p-1}(K)$.

If we have a p-chain, b_p in our complex K, then we say that it is a **p-boundary** if there exists a p+1-chain, c_{p+1}, such that $\partial(c_{p+1}) = b_p$. We call the **p-dimensional boundary group**, $B_p(K)$, as the family of *p-boundaries* formed from the homeomorphic image of $\partial(C_{p+1}(K))$.

Finally we define the **simplicial p-dimensional homology group** of a delta complex, K, as the quotient group $H_p(K) = Z_p(K)/B_p(K)$

3.2 Simplicial Homology Group of the Torus

Let us find the Homology group of the Torus, T. We can start by looking at the triangulation of the Torus induced with an orientation as seen in the diagram below:



In this picture, the point v represents the 0-simplexes for the triangulation of the torus. The lines a, b, c represent the 1-simplexes for the triangulation of the torus, and the triangles U, L represent the 2-simplexes for the triangulation of the torus.

Starting with $H_0(T)$, we find that because we only have one vertex than the kernel of ∂_0, or $Z_0(T)$, ends up being a linear combination with one set of integers which is just $\mathbb{Z}$. Then since we have only one 0-simplex we will get that the $B_0(T) = image(\partial_1) = v - v = 0$ for each 1-simplex. Thus by our previous definition of the homology and quotient group we get $H_0(T) = \mathbb{Z}/0 = \mathbb{Z}$.

Moving onto $H_1(T)$ we follow a similar process. Since we have three 1-simplexes we get that $kernel(\partial_1)$ as the linear combination of three separate integers which ends up giving us $kernel(\partial_1) = \mathbb{Z} \bigoplus \mathbb{Z} \bigoplus \mathbb{Z}$. Then since we only have two 2-

simplexes, the $image(\partial_2)$ is just the addition of two integers, which is just the set $\mathbb{Z}$. So by our previous definition of the homology and quotient group we get that $H_1(T) = \mathbb{Z} \bigoplus \mathbb{Z} \bigoplus \mathbb{Z}/\mathbb{Z} = \mathbb{Z} \bigoplus \mathbb{Z}$

Finally for $H_2(T)$ since we do not have any 3-simplexes then we only need to find $Z_2(T) = ker(\partial_2)$. Once again since we only have two 2-simplexes then we find that the $ker(\partial_2)$ is just the addition of two integers which is just the set $\mathbb{Z}$. So we can define the homology group as $H_2(T) = \mathbb{Z}$

Thus we have shown all the homology groups for the triangulation of the torus in the 2nd dimension.

4 Singular Homology Group

We now move onto a way of extending Homology groups onto more spaces. The simplicial Homology Group relies on a space being composed of properly joined simplexes. The Singular Homology theory finds the Homology groups of more spaces by considering continuous maps from standard simplexes into our space X. The definitions of the singular homology group are quite similar to that of the simplicial homology group. However, the singular homology group gives us the added advantage of being able to find the homology groups of any topological space without knowing the triangulation.

Like with the simplicial homology group we must start by defining a simplex. We define the **unit n-simplex** for a non-negative integer n as a set in $\mathbb{R}^{n+1}$ such that:

$$\Delta_n = \{(x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1} | \Sigma x_i = 1, x_i \geq 0, 0 \leq i \leq n\} \tag{2}$$

We say that v_i is a vertex of Δ_n if its $i-th$ coordinate is 1 and all other coordinates are 0. Note that Δ_n is just the simplex in $\mathbb{R}^{n+1}$ whose vertices are the points $v_0 = (1, 0, ..., 0)$, $v_1 = (0, 1, 0, ..., 0)$, ..., $v_n = (0, 0, ..., 1)$.

We move-on by using functions of the simplex into our space to define our singular homology simplexes. We say the **singular n-simplex** for a space X and a non-negative integer n as a continuous function $s^n : \Delta_n \to X$. We denote the set of all singular n-simplexes in X as, $S_n(X)$. We define the **singular complex** for a space X as $S(X) = \bigcup_{n=0}^{\infty} S_n(X)$

We then continue like in the simplicial homology group by defining a p-chain in terms of singular homology functions. We define the **singular p-chain** for a non-negative integer p as a function $c_p : S_p(X) \to \mathbb{Z}$ from the set of singular p-simplexes in X into the integers such that $c_p(s^p) = 0$ for all but finitely many singular p-simplexes. We say that the set of of all singular p-chains, $C_p(X)$, forms a group under the pointwise operation of addition induced by the integers.

Like in simplicial homology, we must also define what our boundary looks like in singular homology. We define the **singular boundary** for a positive integer p as a homeomorphism $\partial : C_p(X) \to C_{p-1}(X)$ such that

$$\partial(g \cdot s^p) = \Sigma_{i=0}^{p}(-1)g^i \cdot s_i^p \tag{3}$$

where $g \cdot s^p$ represents an elementary singular p-chain, such that g^i is the value of c_p at the singular p-simplex, s_i^p.

We continue by defining a p-cylce and cycle group for singular homology, both concepts were also in the simplicial homology theory. For a positive integer p and space X we then define the **singular p-cycle** as a singlar p-chain, z_p, such that $\partial(z_p) = 0$. We define the **p-dimensional singular cycle group** of X as the set of singular p-cycles that is the kernel of the singular boundary, $\partial : C_p(X) \to C_{p-1}(X)$.

We also will need to define the p-boundary for singular homology in order to find homology groups using a similar method from simplicial homology theory. For a non-negative p we say a singular p-chain, b_p, is a **singular p-boundary** if there is a singular $(p+1)$-chain, c_{p+1}, such that $\partial(c_p + 1) = b_p$. We define the **p-dimensional singular boundary group**, $B_p(X)$, as set of all singular p-boundaries that is the image $\partial(C_{p+1}(X))$.

The actual definition for a singular homology group is the same as with the singular homology group. Finally the **p-dimensional singular homology group** for a set X is the the quotient group:
$H_p(X) = Z_p(X)/B_p(X)$

4.1 Example

Suppose we have a topological space with a single point, $X = x_0$. We start by finding the group of all singular p-chains. Since the generators of the chain group are all the possible p-simplexes that exist in X, which are just maps of the unit p simplexes into X we get that our singular chain group, $C_p(X) = \mathbb{Z}$. Looking at the singular boundaries of we find that $\partial_0(\sigma_0) = 0$ for all 0-simplexes, $\partial_1(\sigma_1) = \sigma_0 - \sigma_0 = 0$ because there is only one σ_0 simplex for X because X contains only one point. Finally $\sigma_2(\sigma_2) = \sigma_1 - \sigma_1 + \sigma_1 = \sigma_1$ which follows from the fact that since X only has one point there is only 1 singular 1-simplex for X. Finally we can easily compute the singular homotopy groups as

$$H_0(X) = Z_0(X)/B_0(X) = ker(\partial_0)/image(\partial_1) = \mathbb{Z}/0 = \mathbb{Z} \tag{4}$$

$$H_1(X) = Z_1(X)/B_1(X) = ker(\partial(1))/image(\partial_2) = \mathbb{Z}/\mathbb{Z} = 0 \tag{5}$$

This is a clear example of a topological space that in which it is not necessary to triangulate the space in order to find its homology groups. The singular homology theory allows us to find homology groups for all toplogical spaces without needing the triangulation of a topological space.

5 Euler Theorem with Homology Groups

One application of Simplicial Homology groups is the Euler-Poincaré Theorem. This theorem allows us to classify polyhedra by their composition in terms of their p-simplexes.

We start by looking at an oriented complex, K. We say a a family $\{z_p^1, z_p^2, ..., z_p^r\}$ of p-cycles is **linearly independent with respect to homology** if there do not exist integers $g_1, ..., g_r$ not all zero such that the chain $\Sigma g_i z_p^i$ is homologous to 0. We say the **p-th Betti Number**, $R_p(K)$, as the largest integer r for which there exists r p-cylces that are linearly independent with respect to homology.

Suppose we have an oriented complex, K, with dimension n, with α_p denoting the number of p-simplexes for K for $p = 0, 1, ..., n$. Then the Euler-Poincaré Theorem tells us the following:

$$\Sigma_{p=0}^n (-1)^p \alpha_p = \Sigma_{p=0}^n (-1)^p R_p(K) \tag{6}$$

Using the Betti Number for a complex K of dimension n, we define the **Euler Characteristic** as

$$\chi(K) = \Sigma_{p=0}^n (-1)^p R_p \tag{7}$$

Euler found a specific application for his formula with *connected planar graphs* as well as for *connected* graphs in general. A graph is **planar** if it is isomorphic to a graph that has been drawn in a plane with no edge-crossings. A graph is **connected** if for every pair of verticies, $a_i, and a_j$, there is a sequence of not necessarily distinct vertices, $a_1, a_2, ..., a_n$ in which a_1 is joined by an edge to a_2, a_2 is joined by an edge to a_3, all the way up to a_{n-1} being joined by an edge to a_n. Finally the **genus** of a graph is defined as the subscript of the first surface among the family of surfaces $S_0, S_1, S_2, ...$ on which the graph can be drawn without edge-crossings

Euler found that if we have a connected planar graph with V vertices, E edges, and F faces then:

$$V - E + F = 2 \tag{8}$$

More generally, if we have a connected graph with the same vertices, edges, and faces, along with a genus G, then :

$$V - E + F = 2 - 2G \tag{9}$$

The importance of these formulas is that they allows us to compare two geometric objects by telling us if they are connected or planar. It is surprisingly difficult to tell if two geometric objects are different especially in higher dimensions. Euler's Theorem allows to compare these objects by simply counting their vertices, edges, faces, and genus. We get more information about objects and their graphs using Euler's formulas.